# Convergence of values in optimal stopping

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**Abstract :** Under the hypothesis of convergence in probability of a sequence of càdlàg processes $(X^n)_n$ to a càdlàg process $X$, we are interested in the convergence of corresponding values in optimal stopping. We give results under hypothesis of inclusion of filtrations or convergence of filtrations.

**Keywords :** Values in optimal stopping, Convergence of stochastic processes, Convergence of filtrations.

## 1 Introduction

Let us consider a càdlàg process $X$. Let us denote by $\mathcal{F}^X$ its natural filtration and by $\mathcal{F}$ the right-continuous associated filtration ($\forall t, \mathcal{F}_t = \mathcal{F}^X_{t^+}$). We denote by $\mathcal{T}_L$ the set of $\mathcal{F}$ stopping times bounded by $L$.

Let $\gamma : [0, +\infty[ \times \mathbb{R} \to \mathbb{R}$ a bounded continuous function. We define the value in optimal stopping of horizon $L$ of the process $X$ by :

$$\Gamma(L) = \sup_{\tau \in \mathcal{T}_L} \mathbb{E}[\gamma(\tau, X_\tau)].$$

**Remark 1** As it is written in Lamberton and Pagès (1990), the value of $\Gamma(L)$ only depends on the law of $X$.

We are interested in the following problem : let us consider a sequence $(X^n)_n$ of processes which converges in probability to a limit process $X$. For all $n$, we denote by $\mathcal{F}^n$ the natural filtration of $X^n$ and by $\mathcal{T}^n_L$ the set of $\mathcal{F}^n$ stopping times bounded by $L$. Then, we define the values in optimal stopping $\Gamma_n(L)$ by $\Gamma_n(L) = \sup\limits_{\tau \in \mathcal{T}^n_L} \mathbb{E}[\gamma(\tau, X^n_\tau)]$. The main aim of this paper is to give conditions under which $(\Gamma_n(L))_n$ converges to $\Gamma(L)$.

In his unpublished manuscript (Aldous, 1981), Aldous proved that if $X$ is quasi-left continuous and if there is extended convergence (in law) of $((X^n, \mathcal{F}^n))_n$ to $(X, \mathcal{F})$, then $(\Gamma_n(L))_n$ converges to $\Gamma(L)$. In their paper (Lamberton and Pagès, 1990), Lamberton and Pagès obtained the same result under the hypothesis of weak extended convergence of $((X^n, \mathcal{F}^n))_n$ to $(X, \mathcal{F})$, quasi-left continuity of the $X^n$'s and Aldous' criterion of tightness for $(X^n)_n$.

As a first step, we are going to prove in section 3 that, under very weak hypothesis, holds the inequality $\Gamma(L) \leqslant \liminf \Gamma_n(L)$.

Then, to prove that $(\Gamma_n(L))_n$ converges to $\Gamma(L)$, it remains to show that $\Gamma(L) \geqslant \limsup \Gamma_n(L)$. This inequality is more difficult and both papers (Aldous, 1981) and (Lamberton and Pagès, 1990) need weak extended convergence to prove it. Here, we prove that it happens under the hypothesis of inclusion of filtrations $\mathcal{F}^n \subset \mathcal{F}$ or under convergence of filtrations.

The main idea in our proof of the inequality $\Gamma(L) \geqslant \limsup \Gamma_n(L)$ is the following. We build a sequence $(\tau^n)$ of $(\mathcal{F}^n)$ stopping times bounded by $L$. Then, we extract a convergent subsequence of $(\tau^n)$ to a random variable $\tau$ and, at the same time, we wish to compare $\mathbb{E}[\gamma(\tau, X_\tau)]$ and $\Gamma(L)$. We are going to do that through two methods.

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First, we will enlarge the space of stopping times, by considering the randomized stopping times and the topology introduced in (Baxter and Chacon, 1977). Baxter and Chacon have shown that the space of randomized stopping times for a right continuous filtration with the associated topology is compact. We are going to use this method in section 5 when we have the inclusion of the filtrations $\mathcal{F}^n \subset \mathcal{F}$ (it means that $\forall t \in [0,T], \mathcal{F}^n_t \subset \mathcal{F}_t$).

When we do not have the previous inclusion, we enlarge the filtration $\mathcal{F}$ associated to the limiting process $X$. This method is used, in a slightly different way, in (Aldous, 1981) and in (Lamberton and Pagès, 1990). In section 6, we enlarge (as little as possible) the limiting filtration so that the limit $\tau^*$ of a convergent subsequence of the randomized $(\mathcal{F}^n)$ stopping times associated to the $(\tau^n)_n$ is a randomized stopping time for this enlarged filtration and we use convergence of filtrations instead of extended convergence.

For technical reasons, we need Aldous' criterion of tightness for the sequence $(X^n)_n$. In section 4, we are going to show that, if $X^n \xrightarrow{\mathbb{P}} X$, Aldous' criterion of tightness for $(X^n)_n$ and quasi-left continuity of the limiting process $X$ are equivalent.

Finally, in section 7, we give applications of the convergence of values in optimal stopping to discretizations and also to financial models.

In what follows, we are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We fix a positive real $T$ and also $L$ between 0 and $T$. Unless otherwise specified, every $\sigma$-field is supposed to be included in $\mathcal{A}$, every process will be indexed by $[0,T]$ and taking values in $\mathbb{R}$ and every filtration will be indexed by $[0,T]$. $\mathbb{D} = \mathbb{D}([0,T])$ denotes the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$. We endow $\mathbb{D}$ with the Skorokhod topology.

For technical background about Skorokhod topology, the reader may refer to (Billingsley, 1999) or (Jacod and Shiryaev, 2002).

# 2 Statement of the result of convergence of the optimal values

The main purpose of this paper is to prove the following Theorem :

**Theorem 2** *Let us consider a càdlàg continuous in probability process $X$ and a sequence $(X^n)_n$ of càdlàg processes. Let $\mathcal{F}$ be the right continuous filtration associated to the natural filtration of $X$ and $(\mathcal{F}^n)_n$ the natural filtrations of the processes $(X^n)_n$. We assume that $X^n \xrightarrow{\mathbb{P}} X$ and that one of the following assertions holds :*
*- for all n, $\mathcal{F}^n \subset \mathcal{F}$,*
*- $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*
*Then, $\Gamma_n(L) \xrightarrow[n\to\infty]{} \Gamma(L)$.*

The notion of convergence of filtrations has been defined in (Hoover, 1991) :

**Definition 3** *We say that $(\mathcal{F}^n)$ converges weakly to $\mathcal{F}$ if for every $A \in \mathcal{F}_T$, $(\mathbb{E}[1_A|\mathcal{F}^n_.])_n$ converges in probability to $\mathbb{E}[1_A|\mathcal{F}_.]$ for Skorokhod topology. We denote $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*

The proof of Theorem 2 will be given through two steps :
- Step 1 : we show that $\Gamma(L) \leqslant \liminf \Gamma_n(L)$,
- Step 2 : we show that $\Gamma(L) \geqslant \limsup \Gamma_n(L)$.

# 3 Proof of the inequality $\Gamma(L) \leqslant \liminf \Gamma_n(L)$

**Theorem 4** *Let us consider a càdlàg process $X$ such that $\mathbb{P}[\Delta X_L \neq 0] = 0$, its natural filtration $\mathcal{F}^X$, a sequence of càdlàg processes $(X^n)_n$ and their natural filtrations $(\mathcal{F}^n)_n$. We suppose that*

$X^n \xrightarrow{\mathbb{P}} X$. *Then* $\Gamma(L) \leqslant \liminf \Gamma_n(L)$.

Proof

The proof is broken in several steps.

**Lemma 5** *Let $\tau$ be a $\mathcal{F}^X$ stopping time bounded by L taking values in a discrete set $\{t_i\}_{i\in I}$ such that $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$, $\forall i$. For all $i$, we consider $A_i = \{\tau = t_i\}$. We define $\tau^n$ by : $\tau^n(\omega) = \min\{t_i : i \in \{j : \mathbb{E}[1_{A_j}|\mathcal{F}^n_{t_j}](\omega) > 1/2\}\}$, $\forall\omega$. Then, $(\tau^n)$ is a sequence of $(\mathcal{T}^n_L)$ such that $(\tau^n, X^n_{\tau^n}) \xrightarrow{\mathbb{P}} (\tau, X_\tau)$.*

Proof

$(\tau^n)_n$ is, by definition, a sequence of $(\mathcal{F}^n)$-stopping times.

Moreover, for all $\omega$, $\tau^n(\omega) \leqslant \max\{t_i, i \in I\} \leqslant L$ because $\tau$ is bounded by $L$. So, $(\tau^n)_n$ is a sequence of $(\mathcal{F}^n)$ stopping times bounded by $L$.

Let us show that $\tau^n \xrightarrow{\mathbb{P}} \tau$.

To prove that, we are going to use the convergence of $\sigma$-fields (also defined in (Hoover, 1991)) :

**Definition 6** *We say that $(\mathcal{A}^n)_n$ converges to $\mathcal{A}$ and denote $\mathcal{A}^n \to \mathcal{A}$ if for every $A \in \mathcal{A}$, $\mathbb{E}[1_A|\mathcal{A}^n] \xrightarrow{\mathbb{P}} 1_A$.*

We have $\mathcal{F}^n_{t_i} \to \mathcal{F}^X_{t_i}, \forall i$ according to the following Lemma :

**Lemma 7** *Let $(X^n)_n$ be a sequence of càdlàg processes that converges in probability to a càdlàg process X, $(\mathcal{F}^n)$ the natural filtrations of the $X^n$'s and $\mathcal{F}^X$ the natural filtration of X. Then, for all t, $\mathcal{F}^n_t \to \mathcal{F}^X_t$.*

Proof

Take $t \in [0,T]$. Let us fix $t_1 < \ldots < t_k \leqslant t$ such that for all $i$, $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$ and let $f : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous.

$X^n \xrightarrow{\mathbb{P}} X$ and for all $i = 1, \ldots, k$, $\mathbb{P}[|\Delta X_{t_i}| \neq 0] = 0$, so

$$(X^n_{t_1}, \ldots, X^n_{t_k}) \xrightarrow{\mathbb{P}} (X_{t_1}, \ldots, X_{t_k}).$$

$f$ is bounded continuous so :

$$f(X^n_{t_1}, \ldots, X^n_{t_k}) \xrightarrow{L^1} f(X_{t_1}, \ldots, X_{t_k}). \tag{1}$$

Take $\varepsilon > 0$.

$$\begin{aligned}
&\mathbb{P}[|\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})|\mathcal{F}^n_t] - f(X_{t_1}, \ldots, X_{t_k})| \geqslant \varepsilon] \\
&\quad\leqslant \quad \mathbb{P}[|\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})|\mathcal{F}^n_t] - \mathbb{E}[f(X^n_{t_1}, \ldots, X^n_{t_k})|\mathcal{F}^n_t]| \geqslant \varepsilon/2] \\
&\qquad\qquad + \mathbb{P}[|\mathbb{E}[f(X^n_{t_1}, \ldots, X^n_{t_k})|\mathcal{F}^n_t] - f(X_{t_1}, \ldots, X_{t_k})| \geqslant \varepsilon/2] \\
&\quad\leqslant \quad \frac{4}{\varepsilon}\mathbb{E}[|f(X^n_{t_1}, \ldots, X^n_{t_k}) - f(X_{t_1}, \ldots, X_{t_k})|] \\
&\qquad\qquad \text{using Markov's inequality} \\
&\quad\xrightarrow[n\to\infty]{} \quad 0 \ \text{ using (1).}
\end{aligned}$$

The conclusion comes with the following characterization of the convergence of $\sigma$-fields, whose proof use exactly same arguments as in the proof of Lemma 3 in (Coquet, Mémin and Słomiński, 2001) :

**Lemma 8** *Let $Y$ be a càdlàg process, $\mathcal{A} = \sigma(\{Y_t, t \geqslant 0\})$ and $(\mathcal{A}^n)$ a sequence of $\sigma$-fields. The following conditions are equivalent :*
*i) $\mathcal{A}^n \to \mathcal{A}$,*
*ii) $\mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_k})|\mathcal{A}^n] \xrightarrow{\mathbb{P}} f(Y_{t_1}, \ldots, Y_{t_k})$ for every continuous bounded function $f : \mathbb{R}^k \to \mathbb{R}$ and $t_1, \ldots, t_k$ continuity points of $Y$.*

Lemma 7 is proved. □

With this Lemma, we can prove the convergence in probability of $(\tau^n)_n$ to $\tau$.
Let us consider a subsequence $(\tau^{\varphi(n)})_n$ of $(\tau^n)_n$. For every $i$, the convergence of the $\sigma$-fields $(\mathcal{F}^n_{t_i})_n$ to $\mathcal{F}^X_{t_i}$ implies $\mathbb{E}[1_{A_i}|\mathcal{F}^{\varphi(n)}_{t_i}] \xrightarrow{\mathbb{P}} 1_{A_i}$. By successive extractions for $i \in I$ finite, there exists $\psi$ such that for every $i$, $\mathbb{E}[1_{A_i}|\mathcal{F}^{\varphi\circ\psi(n)}_{t_i}] \xrightarrow{a.s.} 1_{A_i}$. For $n$ large enough, we have $\tau^{\varphi\circ\psi(n)} = \tau$ *a.s.* Then, $\tau^{\varphi\circ\psi(n)} \xrightarrow{a.s.} \tau$. It follows that $\tau^n \xrightarrow{\mathbb{P}} \tau$.

It remains to show that $X^n_{\tau^n} \xrightarrow{\mathbb{P}} X_\tau$.
$X^n \xrightarrow{\mathbb{P}} X$ so we can find a sequence $(\Lambda^n)_n$ of random time changes such that $\sup_t |X^n_{\Lambda^n(t)} - X_t| \xrightarrow{\mathbb{P}} 0$ and $\sup_t |\Lambda^n(t) - t| \xrightarrow{\mathbb{P}} 0$. Fix $\varepsilon > 0$ and $\eta > 0$. We have :

$$\begin{aligned} &\mathbb{P}[|X^n_{\tau^n} - X_\tau| \geqslant \eta] \\ &\leqslant \quad \mathbb{P}[|X^n_{\tau^n} - X_{(\Lambda^n)^{-1}(\tau^n)}| \geqslant \eta/2] + \mathbb{P}[|X_{(\Lambda^n)^{-1}(\tau^n)} - X_\tau| \geqslant \eta/2]. \end{aligned}$$

There exists $n_0$ such that for every $n \geqslant n_0$, $\mathbb{P}[\sup_t |X^n_{\Lambda^n(t)} - X_t| \geqslant \eta/2] \leqslant \varepsilon$ by choice of $(\Lambda^n)_n$. In particular, for every $n \geqslant n_0$,

$$\mathbb{P}[|X^n_{\tau^n} - X_{(\Lambda^n)^{-1}(\tau^n)}| \geqslant \eta/2] \leqslant \varepsilon. \tag{2}$$

On the other hand, for every $i \in I$ (recall that $I$ is finite), $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$. Then, there exists $\alpha > 0$ such that for every $i \in I$, for every $s$,

$$|s - t_i| \leqslant \alpha \Rightarrow \mathbb{P}[|X_{t_i} - X_s| \geqslant \eta/2] \leqslant \varepsilon. \tag{3}$$

$\tau^n \xrightarrow{\mathbb{P}} \tau$ and $\sup_t |\Lambda^n(t) - t| \xrightarrow{\mathbb{P}} 0$, so $|\tau - (\Lambda^n)^{-1}(\tau^n)| \xrightarrow{\mathbb{P}} 0$. Then, there exists $n_1$ such that for every $n \geqslant n_1$,

$$\mathbb{P}[|\tau - (\Lambda^n)^{-1}(\tau^n)| \geqslant \alpha] \leqslant \varepsilon. \tag{4}$$

Then, for every $n \geqslant n_1$,

$$\begin{aligned} &\mathbb{P}[|X_{(\Lambda^n)^{-1}(\tau^n)} - X_\tau| \geqslant \eta/2] \\ &= \quad \mathbb{P}[|X_{(\Lambda^n)^{-1}(\tau^n)} - X_\tau| 1_{|\tau - (\Lambda^n)^{-1}(\tau^n)| \geqslant \alpha} \geqslant \eta/2] \\ &\qquad + \mathbb{P}[|X_{(\Lambda^n)^{-1}(\tau^n)} - X_\tau| 1_{|\tau - (\Lambda^n)^{-1}(\tau^n)| < \alpha} \geqslant \eta/2] \\ &\leqslant \quad \mathbb{P}[2 \sup_t |X_t| 1_{|\tau - (\Lambda^n)^{-1}(\tau^n)| \geqslant \alpha} \geqslant \eta/2] + \varepsilon \;\; \text{using (3)} \\ &\leqslant \quad \mathbb{P}[|\tau - (\Lambda^n)^{-1}(\tau^n)| \geqslant \alpha] + \varepsilon \\ &\leqslant \quad 2\varepsilon \;\; \text{using (4).} \end{aligned} \tag{5}$$

So, using (2) and (5), for every $n \geqslant \max(n_0, n_1)$,

$$\mathbb{P}[|X^n_{\tau^n} - X_\tau| \geqslant \eta] \leqslant 3\varepsilon.$$

Finally, $(\tau^n, X^n_{\tau^n}) \xrightarrow{\mathbb{P}} (\tau, X_\tau)$.
Lemma 5 is proved. □

With this Lemma, we can prove that Theorem 4 is true for stopping times that takes a finite number of values.

Let us consider a subdivision $\pi$ of $[0, T]$ such that no fixed time of discontinuity of $X$ belongs to $\pi$. We denote by $\mathcal{T}_L^\pi$ the set of $\mathcal{F}$ stopping times taking values in $\pi$ and bounded by $L$. Then, we define :

$$\Gamma^\pi(L) = \sup_{\tau \in \mathcal{T}_L^\pi} \mathbb{E}[\gamma(\tau, X_\tau)].$$

**Lemma 9** $\Gamma^\pi(L) \leqslant \liminf \Gamma_n(L)$.

PROOF
Fix $\varepsilon > 0$. There exists a $\mathcal{F}^X$ stopping time $\tau$ bounded by $L$ taking values in $\pi$ such that

$$\mathbb{E}[\gamma(\tau, X_\tau)] \geqslant \Gamma^\pi(L) - \varepsilon.$$

According to Lemma 5, there exists a sequence $(\tau^n)_n$ of $\mathcal{F}^n$ stopping times bounded by $L$ such that

$$(\tau^n, X^n_{\tau^n}) \xrightarrow{\mathbb{P}} (\tau, X_\tau).$$

$\mathbb{E}[\gamma(\tau^n, X^n_{\tau^n})] \to \mathbb{E}[\gamma(\tau, X_\tau)]$ because $\gamma$ is bounded continuous. Moreover, by definition, for every $n$, $\mathbb{E}[\gamma(\tau^n, X^n_{\tau^n})] \leqslant \Gamma_n(L)$. Il follows that

$$\liminf \mathbb{E}[\gamma(\tau^n, X^n_{\tau^n})] \leqslant \liminf \Gamma_n(L).$$

But, $\liminf \mathbb{E}[\gamma(\tau^n, X^n_{\tau^n})] = \mathbb{E}[\gamma(\tau, X_\tau)] \geqslant \Gamma^\pi(L) - \varepsilon$. So,

$$\Gamma^\pi(L) - \varepsilon \leqslant \liminf \Gamma_n(L), \forall \varepsilon > 0.$$

Then, $\Gamma^\pi(L) \leqslant \liminf \Gamma_n(L)$. □

It remains to link the values of optimal stopping for stopping times taking values in finite subdivisions and $\Gamma(L)$.

**Lemma 10** *Let us consider an increasing sequence $(\pi^k)_k$ of subdivisions without fixed times of continuity of $X$ such that $L \in \pi^k$ for every $k$ (it is possible because $\mathbb{P}[\Delta X_L \neq 0] = 0$) and $|\pi^k| \xrightarrow[k \to +\infty]{} 0$. Then $\Gamma^{\pi^k}(L) \xrightarrow[k \to +\infty]{} \Gamma(L)$.*

PROOF
$(\Gamma^{\pi^k}(L))_k$ is an increasing sequence bounded from above by $\Gamma(L)$. So $(\Gamma^{\pi^k}(L))_k$ converges to a limit $l$ with $l \leqslant \Gamma(L)$. Let us show that $l = \Gamma(L)$.
Fix $\varepsilon > 0$.
We can find $\tau \in \mathcal{T}_L$ such that

$$\mathbb{E}[\gamma(\tau, X_\tau)] \geqslant \Gamma(L) - \varepsilon.$$

We denote $\pi^k = \{t_1^k, \ldots, t_{K_k}^k\}$. Then, let us consider

$$\tau^k = \sum_{i=1}^{K_k - 1} t_{i+1}^k 1_{t_i^k < \tau \leqslant t_{i+1}^k}.$$

For every $k$, $\tau^k \in \mathcal{T}_L^{\pi^k}$ because $\tau$ is bounded by $L$ and $L \in \pi^k$. Since $|\pi^k| \to 0$, we have $\tau^k \xrightarrow{\mathbb{P}} \tau$. Moreover, $\tau^k \geqslant \tau$ and $X$ is right-continuous, so $X_{\tau^k} \xrightarrow{\mathbb{P}} X_\tau$. $\gamma$ is bounded continuous, so

$$\mathbb{E}[\gamma(\tau^k, X_{\tau^k})] \xrightarrow[k \to \infty]{} \mathbb{E}[\gamma(\tau, X_\tau)].$$

But, for every $k$, $\Gamma^{\pi^k}(L) \geqslant \mathbb{E}[\gamma(\tau^k, X_{\tau^k})]$. It follows that

$$l \geqslant \mathbb{E}[\gamma(\tau, X_\tau)] \geqslant \Gamma(L) - \varepsilon.$$

This is true for every $\varepsilon > 0$, so $l \geqslant \Gamma(L)$.
Then, $\Gamma^{\pi^k}(L) \xrightarrow[k \to +\infty]{} \Gamma(L)$ and Lemma 10 si proved. □

At last, Theorem 4 follows from Lemmas 9 and 10. □

**Remark 11** If $\mathbb{P}[\Delta X_L \neq 0] > 0$, the result may not hold any longer. Let us give an example when $L = 1/2$. We consider some processes $x$ and $(x^n)$ defined on $[0,1]$ by $x_t = 1_{[1/2,1]}(t)$ and $x_t^n = 1_{[1/2+1/n,1]}(t)$, $\forall t$. Let us consider $\gamma : [0,+\infty[\times\mathbb{R} \to \mathbb{R}$ such that $\gamma(t,y) = y \wedge 2$. $\gamma$ is a continuous bounded function. We want to compare $\Gamma(1/2)$ and the limit of $\Gamma_n(1/2)$ when $n$ goes to $+\infty$.
We have : $\Gamma(1/2) = \sup\limits_{\tau\in\mathcal{T}_{1/2}} \mathbb{E}[\gamma(\tau, x_\tau)] = \sup\limits_{t\leqslant 1/2} x_t = 1$.
On the other hand, for every $n$, $\Gamma_n(1/2) = \sup\limits_{t\leqslant 1/2} x_t^n = 0$.
So $\liminf \Gamma_n(1/2) = 0 < 1 = \Gamma(1/2)$.

**Remark 12** The Theorem remains true if we replace $\mathcal{F}^X$ by the right continuous filtration associated to $\mathcal{F}$ ($\forall t, \mathcal{F}_t = \mathcal{F}_{t+}^X$) and if we take the $\Gamma(L)$ associated to $\mathcal{F}$.

# 4 Aldous' criterion for tightness

In his papers (Aldous, 1978) and (Aldous, 1989), Aldous deals with the following criterion for tightness :

$$\forall\varepsilon > 0, \lim_{\delta\downarrow 0} \limsup_{n\to+\infty} \sup_{S,T\in\mathcal{T}_L^n, S\leqslant T\leqslant S+\delta} \mathbb{P}[|X_S^n - X_T^n| \geqslant \varepsilon] = 0. \tag{6}$$

He gives many results which links that criterion and weak convergence of sequences of processes.

In his unpublished manuscript (Aldous, 1981), Aldous shows the following result (Corollary 16.23) which links convergence of stopping times to convergence of processes :

**Proposition 13** *Let us consider a sequence of càdlàg processes $(X^n)_n$ that converges in law to a càdlàg process $X$. We denote by $\mathcal{F}^n$ the natural filtrations of the processes $X^n$ and by $\mathcal{F}$ the right continuous natural filtration of the process $X$. Let us consider a sequence $(\tau^n)_n$ of $(\mathcal{F}^n)$-stopping times that converges in law to a random variable $V$. We suppose that we have the join convergence in law of $((\tau^n, X^n))_n$ to $(V, X)$ and that Aldous' criterion for tightness (6) is filled. Then $(\tau^n, X_{\tau^n}^n) \xrightarrow{\mathcal{L}} (V, X_V)$.*

Proof
We just give the sketch of Aldous' proof.

If $\mathbb{P}[\Delta X_V \neq 0] = 0$, using the Skorokhod representation Theorem, we can prove that $(\tau^n, X_{\tau^n}^n) \xrightarrow{\mathcal{L}} (V, X_V)$.

If $\mathbb{P}[\Delta X_V \neq 0] \neq 0$, we can find a decreasing sequence $(\delta_k)_k$ of reals that converges to 0 and such that for every $k$, $\mathbb{P}[\Delta X_{V+\delta_k} \neq 0] = 0$.
Let us take $f : \mathbb{R}^2 \to \mathbb{R}$ bounded and continuous.

$$\begin{aligned}
&|\mathbb{E}[f(\tau^n, X_{\tau^n}^n) - f(V, X_V)]| \\
&\quad\leqslant\ |\mathbb{E}[f(\tau^n, X_{\tau^n}^n) - f(\tau^n + \delta_k, X_{\tau^n+\delta_k}^n)]| \\
&\qquad + |\mathbb{E}[f(\tau^n + \delta_k, X_{\tau^n+\delta_k}^n) - f(V + \delta_k, X_{V+\delta_k})]| \\
&\qquad\ + |\mathbb{E}[f(V + \delta_k, X_{V+\delta_k}) - f(V, X_V)]|.
\end{aligned}$$

But :
- $\forall k$, $\limsup_{n\to+\infty} \mathbb{E}[f(\tau^n + \delta_k, X_{\tau^n+\delta_k}^n) - f(V + \delta_k, X_{V+\delta_k})] = 0$ because $\mathbb{P}[\Delta X_{V+\delta_k} \neq 0] = 0$,
- $\lim_{k\to+\infty} \mathbb{E}[f(V + \delta_k, X_{V+\delta_k}) - f(V, X_V)] = 0$ because $X$ is right-continuous,
- $\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}[f(\tau^{*,n}, X_{\tau^{*,n}}^n) - f(\tau^{*,n} + \delta_k, X_{\tau^{*,n}+\delta_k}^n)] = 0$ using Aldous' criterion.

The result follows. □

**Remark 14** We will see in Proposition 19 an analogous result in the case of randomized stopping times.

The following caracterization of Aldous' Criterion is probably widely known, however I do not know of any reference to a proof of it, so I give one of my own here.

**Proposition 15** *Let us consider a sequence of càdlàg processes* $(X^n)_n$ *and a càdlàg process* $X$ *such that* $X^n \xrightarrow{\mathbb{P}} X$. *The following conditions are equivalent :*
*i)* $X$ *is continuous in probability everywhere, ie for every* $t$ $\mathbb{P}[\Delta X_t \neq 0] = 0$,
*ii) Aldous' criterion for tightness (6) is filled.*

Proof
*i)* $\Rightarrow$ *ii)*. Let $\delta > 0$. Let $(T^n)_n$ and $(S_n)_n$ be two sequences of $\mathcal{T}_L^n$ such that for every $n$, $S^n \leqslant T^n \leqslant S^n + \delta$. Let $\varepsilon > 0$ and $\eta > 0$.
$X^n \xrightarrow{\mathbb{P}} X$ so we can find a sequence of random time changes $(\Lambda^n)_n$ such that $\sup_t |X^n_{\Lambda^n(t)} - X_t| \xrightarrow{\mathbb{P}} 0$. Then there exists $n_0$ such that

$$\forall n \geqslant n_0, \mathbb{P}[\sup_t |X^n_{\Lambda^n(t)} - X_t| \geqslant \eta/3] \leqslant \varepsilon.$$

We have :

$$\begin{aligned}
&\mathbb{P}[|X^n_{S^n} - X^n_{T^n}| \geqslant \eta] \\
&\quad\leqslant \mathbb{P}[|X^n_{S^n} - X_{(\Lambda^n)^{-1}(S^n)}| \geqslant \eta/3] \\
&\qquad + \mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| \geqslant \eta/3] \\
&\qquad + \mathbb{P}[|X_{(\Lambda^n)^{-1}(T^n)} - X^n_{T^n}| \geqslant \eta/3]
\end{aligned} \tag{7}$$

But, for every $n \geqslant n_0$,

$$\mathbb{P}[|X^n_{S^n} - X_{(\Lambda^n)^{-1}(S^n)}| \geqslant \eta/3] \leqslant \mathbb{P}[\sup_t |X^n_{\Lambda^n(t)} - X_t| \geqslant \eta/3] \leqslant \varepsilon, \tag{8}$$

and

$$\mathbb{P}[|X_{(\Lambda^n)^{-1}(T^n)} - X^n_{T^n}| \geqslant \eta/3] \leqslant \mathbb{P}[\sup_t |X^n_{\Lambda^n(t)} - X_t| \geqslant \eta/3] \leqslant \varepsilon. \tag{9}$$

It remains to show that :

$$\lim_{\delta \downarrow 0} \limsup_{n \to +\infty} \mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| \geqslant \eta/3] = 0.$$

$X$ is a càdlàg process, so there exists $\theta > 0$ such that

$$\mathbb{P}[w'(X, \theta) \geqslant \eta/12] \leqslant \varepsilon,$$

where $\forall x \in \mathbb{D}, w'(x, \delta) = \inf_{\{t_i\} \in F_\delta} \max_{1 \leqslant i \leqslant v} w(x, [t_{i-1}, t_i[)$, $F_\delta$ is the set of subdivisions $\{t_i\}_{1 \leqslant i \leqslant v}$ of $[0, T]$ such that $\forall i, t_i - t_{i-1} > \delta$ and $w$ is the modulus of continuity $w(x, [t_{i-1}, t_i[) = \sup\{|x_t - x_s|, t_{i-1} < s < t < t_i\}$ (see e.g. Billingsley, 1999, Section 12).
By defintion of $w'$, there exists a subdivision $\{t_k\}$ such that

$$\forall k, |t_{k+1} - t_k| \geqslant \theta \quad \text{and} \quad \mathbb{P}[\max_k w(X, [t_k, t_{k+1}[) \geqslant \eta/12] \leqslant 2\varepsilon.$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}[|(\Lambda^n)^{-1}(T^n) - (\Lambda^n)^{-1}(T^n + \delta)| \geqslant \theta] \\
&\quad\leqslant \mathbb{P}[|(\Lambda^n)^{-1}(T^n) - T^n| \geqslant \theta/3] + \mathbb{P}[|T^n - (T^n + \delta)| \geqslant \theta/3] \\
&\qquad + \mathbb{P}[|T^n + \delta - (\Lambda^n)^{-1}(T^n + \delta)| \geqslant \theta/3] \\
&\quad\leqslant 2\mathbb{P}[\sup_t |(\Lambda^n)^{-1}(t) - t| \geqslant \theta/3] \quad \text{for every } \delta < \theta/3.
\end{aligned}$$

$\sup_t |(\Lambda^n)^{-1}(t) - t| \xrightarrow{\mathbb{P}} 0$ , so there exists $n_1$ such that

$$\forall n \geqslant n_1, \mathbb{P}[\sup_t |(\Lambda^n)^{-1}(t) - t| \geqslant \theta/3] \leqslant \varepsilon.$$

Then, for every $n \geqslant n_1$, for every $\delta < \theta/3$,

$$\mathbb{P}[|(\Lambda^n)^{-1}(T^n) - (\Lambda^n)^{-1}(S^n)| \geqslant \theta] \leqslant 3\varepsilon. \tag{10}$$

So, for every $n \geqslant n_1$, for every $\delta < \theta/3$,

$$\begin{aligned}
&\mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| \geqslant \eta/3] \\
&\quad = \mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| 1_{|(\Lambda^n)^{-1}(T^n)-(\Lambda^n)^{-1}(S^n)|<\theta} \geqslant \eta/3] \\
&\qquad + \mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| 1_{|(\Lambda^n)^{-1}(T^n)-(\Lambda^n)^{-1}(S^n)|\geqslant\theta} \geqslant \eta/3].
\end{aligned}$$

But,

$$\begin{aligned}
&\mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| 1_{|(\Lambda^n)^{-1}(T^n)-(\Lambda^n)^{-1}(S^n)|<\theta} \geqslant \eta/3] \\
&\quad \leqslant \mathbb{P}[(2 \max_k w(X, [t_k, t_{k+1}[) + \max_k |\Delta X_{t_k}|) \geqslant \eta/3] \\
&\quad \leqslant \mathbb{P}[\max_k w(X, [t_k, t_{k+1}[) \geqslant \eta/12] + \mathbb{P}[\max_k |\Delta X_{t_k}| \geqslant \eta/6] \\
&\quad \leqslant 2\varepsilon + \sum_k \mathbb{P}[|\Delta X_{t_k}| \geqslant \eta/6] \\
&\quad \leqslant 2\varepsilon \ \text{ because } X \text{ has no fixed time of discontinuity}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| 1_{|(\Lambda^n)^{-1}(T^n)-(\Lambda^n)^{-1}(S^n)|\geqslant\theta} \geqslant \eta/3] \\
&\quad \leqslant \mathbb{P}[2 \sup_t |X_t| 1_{|(\Lambda^n)^{-1}(T^n)-(\Lambda^n)^{-1}(S^n)|\geqslant\theta} \geqslant \eta/3] \\
&\quad \leqslant \mathbb{P}[|(\Lambda^n)^{-1}(T^n) - (\Lambda^n)^{-1}(S^n)| \geqslant \theta] \\
&\quad \leqslant 3\varepsilon \ \text{ using (10).}
\end{aligned}$$

So for every $n \geqslant n_1$, for every $\delta < \theta/3$,

$$\mathbb{P}[|X_{(\Lambda^n)^{-1}(S^n)} - X_{(\Lambda^n)^{-1}(T^n)}| \geqslant \eta/3] \leqslant 5\varepsilon. \tag{11}$$

Finally, using inequalities (7), (8), (9) and (11), for every $n \geqslant \max(n_0, n_1)$, for every $\delta < \theta/3$,

$$\mathbb{P}[|X^n_{S^n} - X^n_{T^n}| \geqslant \eta] \leqslant 7\varepsilon.$$

$n_0$, $n_1$ and $\theta$ do not depend on $(T_n)_n$ and $(S_n)_n$. Then, for every $n \geqslant \max(n_0, n_1)$, for every $\delta < \theta/3$,

$$\sup_{S,T \in \mathcal{T}^n_L, S \leqslant T \leqslant S+\delta} \mathbb{P}[|X^n_S - X^n_T| \geqslant \eta] \leqslant 7\varepsilon.$$

Aldous' criterion follows.

*ii)* $\Rightarrow$ *i)*. We suppose that there exists $t_0$ such that $\mathbb{P}[\Delta X_{t_0} \neq 0] > 0$.
Let $\varepsilon > 0$ and $\eta > 0$ be such that $\mathbb{P}[|\Delta X_{t_0}| \geqslant 2\varepsilon] \geqslant 2\eta$.
$X^n \xrightarrow{\mathbb{P}} X$ so we can find a random sequence $(t^n)_n$ such that $t^n \xrightarrow{\mathbb{P}} t_0$ and $\Delta X^n_{t^n} \xrightarrow{\mathbb{P}} \Delta X_{t_0}$. There exists $n_0$ such that for every $n \geqslant n_0$,

$$\mathbb{P}[|t^n - t_0| \geqslant \delta/2] \leqslant \eta/2 \ \text{ and } \ \mathbb{P}[|\Delta X^n_{t^n} - \Delta X_{t_0}| \geqslant \varepsilon] \leqslant \eta/2. \tag{12}$$

We are going to show that for every $n \geqslant n_0$, for $\delta$ large enough,

$$\mathbb{P}[|X^n_{t_0+\delta/2} - X^n_{t_0-\delta/2}| \geqslant \varepsilon/3] \geqslant \eta/2.$$

Then, for every $n \geqslant n_0$,

$$\begin{aligned}
&\mathbb{P}[|\Delta X^n_{t^n}| \geqslant \varepsilon] \\
&\quad = \mathbb{P}[|\Delta X^n_{t^n}| 1_{|t^n - t_0| \geqslant \delta/2} \geqslant \varepsilon] + \mathbb{P}[|\Delta X^n_{t^n}| 1_{|t^n - t_0| < \delta/2} \geqslant \varepsilon] \\
&\quad \leqslant \mathbb{P}[|t^n - t_0| \geqslant \delta/2] + \mathbb{P}[|X^n_{t^n} - X^n_{t_0+\delta/2}| 1_{|t^n - t_0| < \delta/2} \geqslant \varepsilon/3] \\
&\qquad + \mathbb{P}[|X^n_{t_0+\delta/2} - X^n_{t_0-\delta/2}| \geqslant \varepsilon/3] + \mathbb{P}[|X^n_{t_0-\delta/2} - X^n_{t^n-}| 1_{|t^n - t_0| < \delta/2} \geqslant \varepsilon/3]
\end{aligned} \tag{13}$$

$(X^n)_n$ is tight. So, we can find $\delta_0 > 0$ and $n_1 \in \mathbb{N}$ such that for every $\delta \leqslant \delta_0$, for every $n \geqslant n_1$,

$$\mathbb{P}[w'(X^n, \delta) \geqslant \varepsilon/6] \leqslant \eta/6.$$

Then, we can find a finite subdivision $\{t_k\}$ such that

$$\forall k, t_{k+1} - t_k \geqslant \delta \quad \text{and} \quad \mathbb{P}[\max_k w(X^n, [t_k, t_{k+1}[) \geqslant \varepsilon/3] \leqslant \eta/4.$$

We know that for every $n \geqslant \max(n_0, n_1)$,

$$\begin{aligned} 2\eta &\leqslant \mathbb{P}[|\Delta X_{t_0}| \geqslant 2\varepsilon] \\ &\leqslant \mathbb{P}[|\Delta X^n_{t^n} - \Delta X_{t_0}| \geqslant \varepsilon] + \mathbb{P}[|\Delta X^n_{t^n}| \geqslant \varepsilon] \\ &\leqslant \eta/2 + \mathbb{P}[|\Delta X^n_{t^n}| \geqslant \varepsilon]. \end{aligned}$$

In particular, for every $n \geqslant \max(n_0, n_1)$,

$$\mathbb{P}[|\Delta X^n_{t^n}| \geqslant \varepsilon] \geqslant 3\eta/2. \tag{14}$$

So, for every $n \geqslant \max(n_0, n_1)$, $t^n \in \{t_k\}$.
Then, for every $\delta \leqslant \delta_0$, for every $n \geqslant \max(n_0, n_1)$,

$$\mathbb{P}[|X^n_{t^n} - X^n_{t_0+\delta/2}| 1_{|t^n - t_0| < \delta/2} \geqslant \varepsilon/3] \leqslant \mathbb{P}[\max_k w(X^n, [t_k, t_{k+1}[) \geqslant \varepsilon/3] \leqslant \eta/4. \tag{15}$$

On the same way,

$$\mathbb{P}[|X^n_{t_0-\delta/2} - X^n_{t^n-}| 1_{|t^n - t_0| < \delta/2} \geqslant \varepsilon/3] \leqslant \eta/4. \tag{16}$$

Finally, using (13) and inequalities (12), (14), (15) and (16), for every $\delta \leqslant \delta_0$, for every $n \geqslant \max(n_0, n_1)$,

$$3\eta/2 \leqslant \mathbb{P}[|\Delta X^n_{t^n}| \geqslant \varepsilon] \leqslant \eta/2 + \eta/4 + \mathbb{P}[|X^n_{t_0+\delta/2} - X^n_{t_0-\delta/2}| \geqslant \varepsilon/3] + \eta/4.$$

So, for every $\delta \leqslant \delta_0$, for every $n \geqslant \max(n_0, n_1)$,

$$\begin{aligned} \eta/2 &\leqslant \mathbb{P}[|X^n_{t_0+\delta/2} - X^n_{t_0-\delta/2}| \geqslant \varepsilon/3] \\ &\leqslant \sup_{S,T \in \mathcal{T}^n_L, S \leqslant T \leqslant S+\delta} \mathbb{P}[|X^n_{T^n+\delta} - X^n_{T^n}| \geqslant \varepsilon/3]. \end{aligned}$$

Taking the lim sup when $n$ tends to infinity and the limit when $\delta$ decreases to 0, we have :

$$\eta/2 \leqslant \lim_{\delta \downarrow 0} \limsup_{n \to +\infty} \sup_{S,T \in \mathcal{T}^n_L, S \leqslant T \leqslant S+\delta} \mathbb{P}[|X^n_{T^n+\delta} - X^n_{T^n}| \geqslant \varepsilon/3],$$

which is in contradiction with Aldous' criterion. The result follows. □

# 5 Proof of the inequality $\Gamma(L) \geqslant \limsup \Gamma_n(L)$ when for every $n$, $\mathcal{F}^n \subset \mathcal{F}$

## 5.1 Randomized stopping times

The notion of randomized stopping times has been introduced in (Baxter and Chacon, 1977) and this notion has been used in (Meyer, 1978) under the french name "temps d'arrêt flous".

We are given a filtration $\mathcal{F}$. Let us denote by $\mathcal{B}$ the Borel $\sigma$-field on $[0, 1]$. Then, we define the filtration $\mathcal{G}$ on $\Omega \times [0, 1]$ such that $\forall t$, $\mathcal{G}_t = \mathcal{F}_t \times \mathcal{B}$. A map $\tau : \Omega \times [0, 1] \to [0, +\infty]$ is called a randomized $\mathcal{F}$ stopping time if $\tau$ is a $\mathcal{G}$ stopping time. We denote by $\mathcal{T}^*$ the set of randomized stopping times and by $\mathcal{T}^*_L$ the set of randomized stopping times bounded by $L$. $\mathcal{T}$ is included in $\mathcal{T}^*$ and the application $\tau \mapsto \tau^*$, where $\tau^*(\omega, t) = \tau(\omega)$ for every $\omega$ and every $t$, maps $\mathcal{T}$ into

$\mathcal{T}^*$. In the same way, $\mathcal{T}_L$ is included in $\mathcal{T}_L^*$.

On the space $\Omega \times [0,1]$, we put the probability measure $\mathbb{P} \otimes \mu$ where $\mu$ is Lebesgue's measure on $[0,1]$. In their paper (Baxter and Chacon, 1977), Baxter and Chacon define the convergence of randomized stopping times by the following :

$$\tau^{*,n} \xrightarrow{BC} \tau^* \text{ iff } \forall f \in \mathcal{C}_b([0,\infty]), \forall Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E}[Yf(\tau^{*,n})] \to \mathbb{E}[Yf(\tau^*)],$$

where $\mathcal{C}_b([0,\infty])$ is the set of bounded continuous functions on $[0,\infty]$.

Taking $Y = 1$, we note that this convergence implies the "usual" convergence in law.

This notion is a particular case of "stable convergence" introduced in (Renyi, 1963) and studied in (Jacod and Mémin, 1981). This is the link between convergence in probability and stable convergence that we are going to use :

**Lemma 16** *Let us consider a sequence $(\tau^n)_n$ of $\mathcal{F}$ stopping times that converges in probability to $\tau$. Then the sequence $(\tau^{*,n})_n$ where $\tau^{*,n}(\omega,t) = \tau^n(\omega)$ $\forall\omega$, $\forall t$, converges in Baxter and Chacon's way to $\tau^*$ where $\tau^*(\omega,t) = \tau(\omega)$, $\forall\omega$, $\forall t$.*

One of the main interests of this notion is, as it is shown in (Baxter and Chacon, 1977, Theorem 1.5), that the set of randomized stopping times for a right continuous filtration is compact for Baxter and Chacon's topology.

The following Proposition is the main argument in the proof of Theorem 22 below.

**Proposition 17** *Let us consider a sequence of filtrations $(\mathcal{F}^n)$ and a right continuous filtration $\mathcal{F}$ such that $\forall n$, $\mathcal{F}^n \subset \mathcal{F}$. Let $(\tau^n)_n$ be a sequence of $(\mathcal{T}_L^n)_n$. Then, there exists a randomized $\mathcal{F}$ stopping time $\tau^*$ and a subsequence $(\tau^{\varphi(n)})_n$ such that $\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*$ where for every $n$, $\tau^{*,n}(\omega,t) = \tau^n(\omega)$ $\forall\omega$, $\forall t$.*

Proof

For every $n$, $\mathcal{F}^n \subset \mathcal{F}$, $(\tau^n)_n$ is a sequence of $\mathcal{F}$ stopping times so, by definition, $(\tau^{*,n})_n$ is a sequence of randomized $\mathcal{F}$ stopping times. According to (Baxter and Chacon, 1977, Theorem 1.5), we can find a randomized $\mathcal{F}$ stopping time $\tau^*$ and a subsequence $(\tau^{\varphi(n)})_n$ such that $\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*$. □

Now, we define $X_{\tau^*}$ by $X_{\tau^*}(\omega,v) = X_{\tau^*(\omega,v)}(\omega)$, for every $(\omega,v) \in \Omega \times [0,1]$. Then, we can prove the following Lemma :

**Lemma 18** *Let us consider $\Gamma^*(L) = \sup_{\tau^* \in \mathcal{T}_L^*} \mathbb{E}[\gamma(\tau^*, X_{\tau^*})]$. Then $\Gamma^*(L) = \Gamma(L)$.*

Proof

- $\mathcal{T}_L$ is included $\mathcal{T}_L^*$. Then $\Gamma(L) \leqslant \Gamma^*(L)$.
- Let $\tau^* \in \mathcal{T}_L^*$. We consider, for every $v$, $\tau_v(\omega) = \tau^*(\omega,v), \forall\omega$.

For every $v \in [0,1]$, for every $t \in [0,T]$,

$$\{\omega : \tau_v(\omega) \leqslant t\} \times \{v\} = \{(\omega,x) : \tau^*(\omega,x) \leqslant t\} \cap (\Omega \times \{v\}).$$

But, $\{(\omega,x) : \tau^*(\omega,x) \leqslant t\} \in \mathcal{F}_t \times \mathcal{B}$ because $\tau^*$ is a randomized $\mathcal{F}$ stopping time and $\Omega \times \{v\} \in \mathcal{F}_t \times \mathcal{B}$. So, $\{\omega : \tau_v(\omega) \leqslant t\} \times \{v\} \in \mathcal{F}_t \times \mathcal{B}$. Consequently,

$$\{\omega : \tau_v(\omega) \leqslant t\} \in \mathcal{F}_t.$$

Then, for every $v$, $\tau_v$ is a $\mathcal{F}$ stopping time bounded by $L$. We have :

$$\begin{aligned}
\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] &= \int_\Omega \int_0^1 \gamma(\tau^*(\omega,v), X_{\tau^*(\omega,v)}(\omega))d\mathbb{P}(\omega)dv \\
&= \int_0^1 \left( \int_\Omega \gamma(\tau^*(\omega,v), X_{\tau_v(\omega)}(\omega))d\mathbb{P}(\omega) \right) dv \\
&= \int_0^1 \mathbb{E}[\gamma(\tau_v, X_{\tau_v})]dv \\
&\leqslant \Gamma(L) \text{ because, for every } v,\ \tau_v \in \mathcal{T}_L.
\end{aligned}$$

Taking the sup for $\tau^*$ in $\mathcal{T}_L^*$, we get $\Gamma^*(L) \leqslant \Gamma(L)$.
Lemma 18 is proved. □

We have an analogous of Proposition 13 in the setting of randomized stopping times.

**Proposition 19** *Let us consider a sequence $(X^n)_n$ of càdlàg processes that converges in law to a càdlàg process $X$, $\mathcal{F}^n$ the natural filtrations of the $X^n$'s and $\mathcal{F}$ the right continuous natural filtration of the process $X$. Let $(\tau^n)_n$ be a sequence of $(\mathcal{F}^n)$ stopping times such that the associated sequence $(\tau^{*,n})_n$ of randomized stopping times ($\tau^{*,n}(\omega,t) = \tau^n(\omega)\ \forall\omega,\ \forall t$) converges in law to a random variable $V$. We suppose that $(\tau^{*,n}, X^n) \xrightarrow{\mathcal{L}} (V,X)$ and that Aldous' criterion 6 is filled. Then $(\tau^{*,n}, X^n_{\tau^{*,n}}) \xrightarrow{\mathcal{L}} (V, X_V)$.*

Proof
The proof of Proposition 19 follows the lines of the proof of (Aldous, 1981, Corollary 16.23) (Proposition 13 in this paper). □

**Remark 20** We point out that, in this Proposition, Aldous' Criterion is filled by the original -not randomized- stopping times.

When $X^n \xrightarrow{\mathbb{P}} X$ and when $(\tau^{*,n})_n$ is a sequence of randomized stopping times converging in the sense of Baxter and Chacon to a random variable $\tau^*$, we have the join convergence in law of $((X^n, \tau^{*,n}))_n$ to $(X, \tau^*)$ :

**Proposition 21** *Let us consider a sequence $(X^n)_n$ of càdlàg processes converging in probability to a càdlàg process $X$, $\mathcal{F}^n$ the natural filtrations of the $X^n$'s and $\mathcal{F}$ the right continuous natural filtration of the process $X$. Let $(\tau^{*,n})_n$ be a sequence of randomized $(\mathcal{F}^n)$ stopping times converging to the randomized stopping time $\tau$ under Baxter and Chacon's topology.*
*Then $(X^n, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X, \tau^*)$.*

Proof
- As $(X^n)_n$ and $(\tau^{*,n})_n$ are tight, $((X^n, \tau^{*,n}))_n$ is tight.
- We are now going to identify the limit thanks to the finite-dimensional convergence.
Let $k \in \mathbb{N}$ and $t_1 < \ldots < t_k$ such that for every $i$, $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$. Let us show that $(X^n_{t_1}, \ldots, X^n_{t_k}, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X_{t_1}, \ldots, X_{t_k}, \tau^*)$.
In a first time, let us consider $f : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ bounded continuous.

$$\begin{aligned}
&|\mathbb{E}[f(X^n_{t_1}, \ldots, X^n_{t_k})g(\tau^n)] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)]| \\
&\quad\leqslant |\mathbb{E}[(f(X^n_{t_1}, \ldots, X^n_{t_k}) - f(X_{t_1}, \ldots, X_{t_k}))g(\tau^{*,n})]| \\
&\qquad + |\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)]| \\
&\quad\leqslant \|g\|_\infty \mathbb{E}[|f(X^n_{t_1}, \ldots, X^n_{t_k}) - f(X_{t_1}, \ldots, X_{t_k})|] \\
&\qquad + |\mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \ldots, X_{t_k})g(\tau^*)]|
\end{aligned}$$

But, $X^n \xrightarrow{\mathbb{P}} X$ and for every $i$, $\mathbb{P}[\Delta X_{t_i} \neq 0] = 0$ so $(X^n_{t_1}, \dots, X^n_{t_k}) \xrightarrow{\mathbb{P}} (X_{t_1}, \dots, X_{t_k})$. Moreover, $f$ is bounded continuous, so

$$\mathbb{E}[|f(X^n_{t_1}, \dots, X^n_{t_k}) - f(X_{t_1}, \dots, X_{t_k})|] \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, by definition of Baxter and Chacon's convergence,

$$\mathbb{E}[f(X_{t_1}, \dots, X_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \dots, X_{t_k})g(\tau^*)] \xrightarrow[n\to+\infty]{} 0.$$

Then,

$$\mathbb{E}[f(X^n_{t_1}, \dots, X^n_{t_k})g(\tau^{*,n})] - \mathbb{E}[f(X_{t_1}, \dots, X_{t_k})g(\tau^*)] \xrightarrow[n\to+\infty]{} 0.$$

Let us now consider $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}$ continuous and bounded.
Let us fix $\varepsilon > 0$.
$((X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n}))_n$ is tight. We can find a compact set $K_\varepsilon$ such that

$$\mathbb{P}[(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n}) \notin K_\varepsilon] \leqslant \varepsilon. \tag{17}$$

We write $\varphi = \varphi 1_{K_\varepsilon} + \varphi 1_{K^c_\varepsilon}$.
$\varphi 1_{K_\varepsilon}$ is a continuous function on the compact set $K_\varepsilon$. Using Weierstrass' Theorem, we can find a polynomial function $P$ such that

$$\|\varphi 1_{K_\varepsilon} - P 1_{K_\varepsilon}\|_\infty \leqslant \varepsilon. \tag{18}$$

Using the previous result and the linearity of expectation, we have

$$\mathbb{E}[(P1_{K_\varepsilon})(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[(P1_{K_\varepsilon})(X_{t_1}, \dots, X_{t_k}, \tau^*)] \xrightarrow[n\to+\infty]{} 0. \tag{19}$$

Finally,

$$\begin{aligned}
&|\mathbb{E}[\varphi(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[\varphi(X_{t_1}, \dots, X_{t_k}, \tau^*)]| \\
&\quad = \quad |\mathbb{E}[(\varphi(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[\varphi(X_{t_1}, \dots, X_{t_k}, \tau^*))1_{K_\varepsilon}(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})]| \\
&\qquad\quad +|\mathbb{E}[(\varphi(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[\varphi(X_{t_1}, \dots, X_{t_k}, \tau^*))1_{K^c_\varepsilon}(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})]| \\
&\quad \leqslant \quad 2\|\varphi 1_{K_\varepsilon} - P1_{K_\varepsilon}\|_\infty \\
&\qquad\qquad +\mathbb{E}[(P1_{K_\varepsilon})(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[(P1_{K_\varepsilon})(X_{t_1}, \dots, X_{t_k}, \tau^*)] \\
&\qquad\qquad\quad +\|\varphi\|_\infty \mathbb{P}[(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n}) \notin K_\varepsilon] \\
&\quad \leqslant \quad \mathbb{E}[(P1_{K_\varepsilon})(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[(P1_{K_\varepsilon})(X_{t_1}, \dots, X_{t_k}, \tau^*)] \\
&\qquad\qquad +(2 + \|\varphi\|_\infty)\varepsilon \;\text{ using (17) and (18).}
\end{aligned}$$

Taking the limit for $n$, using (19), we obtain :

$$\limsup_n |\mathbb{E}[\varphi(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] - \mathbb{E}[\varphi(X_{t_1}, \dots, X_{t_k}, \tau^*)]| \leqslant (2 + \|\varphi\|_\infty)\varepsilon.$$

This is true for every $\varepsilon > 0$, so we have

$$\mathbb{E}[\varphi(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n})] \xrightarrow[n\to+\infty]{} \mathbb{E}[\varphi(X_{t_1}, \dots, X_{t_k}, \tau^*)].$$

Then, $(X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X_{t_1}, \dots, X_{t_k}, \tau^*)$.

The tightness of the sequence $((X^n, \tau^{*,n}))_n$ and the finite-dimensional convergence on a dense set to $(X, \tau^*)$ implies $(X^n, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X, \tau^*)$. □

## 5.2 Application to the proof of the inequality $\limsup \Gamma_n(L) \leqslant \Gamma(L)$

We can now prove a result about the convergence of optimal values.

**Theorem 22** *Let us consider a càdlàg process $X$ continuous in probability, its natural right continuous filtration $\mathcal{F}$, a sequence $(X^n)_n$ of càdlàg processes and their natural filtrations $(\mathcal{F}^n)_n$. We suppose that $X^n \xrightarrow{\mathbb{P}} X$ and $\forall n$, $\mathcal{F}^n \subset \mathcal{F}$.*
*Then $\limsup \Gamma_n(L) \leqslant \Gamma(L)$.*

Proof
There exists a subsequence $(\Gamma_{\varphi(n)}(L))_n$ converging to $\limsup \Gamma_n(L)$.
Let us fix $\varepsilon > 0$. We can find a sequence $(\tau^{\varphi(n)})_n$ of $(\mathcal{T}_L^{\varphi(n)})_n$ such that

$$\forall n, \mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \geqslant \Gamma_{\varphi(n)}(L) - \varepsilon.$$

We consider the sequence $(\tau^{*,n})_n$ of randomized stopping times associated to $(\tau^n)_n$ : for every $n$, $\tau^{*,n}(\omega, t) = \tau^n(\omega)$, $\forall \omega$, $\forall t$.
$\mathcal{F}^n \subset \mathcal{F}$ and $(\tau^{\varphi(n)})$ is a sequence of $(\mathcal{F}^{\varphi(n)})_n$-stopping times bounded by $L$, so using Proposition 17, there exists a randomized $\mathcal{F}$ stopping time $\tau^*$ and a subsequence $(\tau^{\varphi\circ\psi(n)})$ such that $\tau^{*,\varphi\circ\psi(n)} \xrightarrow{BC} \tau^*$.
$X^{\varphi\circ\psi(n)} \xrightarrow{\mathbb{P}} X$ and $\tau^{*,\varphi\circ\psi(n)} \xrightarrow{BC} \tau^*$, so using Proposition 21,

$$(X^{\varphi\circ\psi(n)}, \tau^{*,\varphi\circ\psi(n)}) \xrightarrow{\mathcal{L}} (X, \tau^*).$$

Then, using Proposition 19, we have :

$$(\tau^{*,\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{*,\varphi\circ\psi(n)}}) \xrightarrow{\mathcal{L}} (\tau^*, X_{\tau^*}).$$

Since $\gamma$ is continuous and bounded, we have :

$$\mathbb{E}[\gamma(\tau^{*,\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{*,\varphi\circ\psi(n)}})] \to \mathbb{E}[\gamma(\tau^*, X_{\tau^*})].$$

But, $\mathbb{E}[\gamma(\tau^{*,\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{*,\varphi\circ\psi(n)}})] = \mathbb{E}[\gamma(\tau^{\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{\varphi\circ\psi(n)}})]$ by definition of $(\tau^{*,n})$ and by choice of $\varphi$, $\mathbb{E}[\gamma(\tau^{\varphi\circ\psi(n)}, X^{\varphi\circ\psi(n)}_{\tau^{\varphi\circ\psi(n)}})] \geqslant \Gamma_{\varphi\circ\psi(n)}(L) - \varepsilon$. So,

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_{\varphi\circ\psi(n)}(L) - \varepsilon.$$

By selection of $\varphi$, $\limsup \Gamma_{\varphi\circ\psi(n)}(L) = \limsup \Gamma_n(L)$. Then,

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_n(L) - \varepsilon.$$

This is true for every $\varepsilon > 0$, so

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_n(L).$$

But, by definition, $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \leqslant \Gamma^*(L)$ because $\tau^*$ is a randomized stopping time. As $\Gamma^*(L) = \Gamma(L)$ by Lemma 18, we deduce $\Gamma(L) \geqslant \limsup \Gamma_n(L)$. □

**Remark 23** In the previous Theorem, the most important argument is that we know things about the nature of the limit of the subsequence of stopping times thanks to Proposition 17. If we remove the inclusion of the filtrations $\mathcal{F}^n \subset \mathcal{F}, \forall n$, the limit of the subsequence is no longer a randomized $\mathcal{F}$ stopping time. In this case, we can't compare $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})]$ and $\Gamma^*(L)$.

# 6 Proof of the inequality $\limsup \Gamma_n(L) \leqslant \Gamma(L)$ when $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$

**Theorem 24** *Let us consider a sequence of càdlàg processes $(X^n)_n$, their natural filtrations $(\mathcal{F}^n)_n$, a càdlàg process continuous in probability $X$ and its right continuous natural filtration $\mathcal{F}$. We suppose $X^n \xrightarrow{\mathbb{P}} X$ and $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*
*Then* $\limsup \Gamma_n(L) \leqslant \Gamma(L)$.

Proof
We argue more or less as Aldous in the second part of the proof of (Aldous, 1981, Theorem 17.2).
We can find a subsequence $(\Gamma_{\varphi(n)}(L))_n$ converging to $\limsup \Gamma_n(L)$.
Let us take $\varepsilon > 0$. There exists a sequence $(\tau^{\varphi(n)})_n$ of $(\mathcal{T}_L^{\varphi(n)})_n$ such that

$$\forall n, \mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \geqslant \Gamma_{\varphi(n)}(L) - \varepsilon.$$

Let us consider the sequence $(\tau^{*,n})_n$ of associated randomized $(\mathcal{F}^n)$ stopping times like in 5.1. Taking the filtration $\mathcal{H} = (\bigvee_n \mathcal{F}^n) \vee \mathcal{F}$, $(\tau^{*,n})$ is a bounded sequence of randomized $\mathcal{H}$ stopping times. Then, using (Baxter and Chacon, 1977, Theorem 1.5), we can find an increasing map $\varphi$ and a randomized $\mathcal{H}$ stopping time $\tau^*$ ($\tau^*$ is not a priori a randomized $\mathcal{F}$ stopping time) such that

$$\tau^{*,\varphi(n)} \xrightarrow{BC} \tau^*.$$

Using Proposition 21, we obtain $(X^{\varphi(n)}, \tau^{*,\varphi(n)}) \xrightarrow{\mathcal{L}} (X, \tau^*)$. Then, with Proposition 19, we have $(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}}) \xrightarrow{\mathcal{L}} (\tau^*, X_{\tau^*})$. So,

$$\mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \xrightarrow[n \to +\infty]{} \mathbb{E}[\gamma(\tau^*, X_{\tau^*})].$$

On the other hand, $\mathbb{E}[\gamma(\tau^{\varphi(n)}, X^{\varphi(n)}_{\tau^{\varphi(n)}})] \geqslant \Gamma_{\varphi(n)}(L) - \varepsilon$. So, when $n$ tends to infinity, it results :

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_n(L) - \varepsilon.$$

This is true for every $\varepsilon > 0$, hence we have

$$\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \geqslant \limsup \Gamma_n(L). \tag{20}$$

It remains to compare $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})]$ and $\Gamma(L)$.
Let us consider the smaller right continuous filtration $\mathcal{G}$ such that $X$ is $\mathcal{G}$ adapted and $\tau^*$ is a randomized $\mathcal{G}$ stopping time. It is clear that $\mathcal{F} \subset \mathcal{G}$. For every $t$, we have

$$\mathcal{G}_t \times \mathcal{B} = \cap_{s>t} \sigma(A \times [0,1], \{\tau^* \leqslant u\}, A \in \mathcal{F}_s, u \leqslant s).$$

We consider the set $\tilde{\mathcal{T}}_L$ of randomized $\mathcal{G}$ stopping times bounded by $L$ and we define $\tilde{\Gamma}(L) = \sup_{\tilde{\tau} \in \tilde{\mathcal{T}}_L} \mathbb{E}[\gamma(\tilde{\tau}, X_{\tilde{\tau}})]$.

By definition of $\mathcal{G}$, $\tau^* \in \tilde{\mathcal{T}}_L$ so $\mathbb{E}[\gamma(\tau^*, X_{\tau^*})] \leqslant \tilde{\Gamma}(L)$.

We are going to end the proof using the following Lemma, that is an adaptation of (Lamberton and Pagès, 1990, Proposition 3.5) to our enlargement of filtration :

**Lemma 25** *If $\mathcal{G}_t \times \mathcal{B}$ and $\mathcal{F}_T \times \mathcal{B}$ are conditionally independent given $\mathcal{F}_t \times \mathcal{B}$ for every $t \in [0,T]$, then $\tilde{\Gamma}(L) = \Gamma^*(L)$.*

Proof
The proof is the same as the proof of (Lamberton and Pagès, 1990, Proposition 3.5) with $(\mathcal{F}_t \times \mathcal{B})_{t \in [0,T]}$ and $(\mathcal{G}_t \times \mathcal{B})_{t \in [0,T]}$ instead of $\mathcal{F}^Y$ and $\mathcal{F}$ and with the process $X^*$ such that for every $\omega$, for every $v \in [0,1]$, for every $t \in [0,T]$, $X^*_t(\omega, v) = X_t(\omega)$ instead of the process $Y$. $\square$

According to (Brémaud and Yor, 1978, Theorem 3), the condition of conditional independence required in Lemma 25 is equivalent to the following assumption :

$$\forall t \in [0,T], \forall Z \in L^1(\mathcal{F}_T \times \mathcal{B}), \mathbb{E}[Z|\mathcal{F}_t \times \mathcal{B}] = \mathbb{E}[Z|\mathcal{G}_t \times \mathcal{B}]. \tag{21}$$

We will show that the assumptions of Theorem 24 imply those of Lemma 25, therefore proving inequality (21).
Note that in (Aldous, 1981) and in (Lamberton and Pagès, 1990), they need extended convergence to prove (21).

Without loss of generality, we suppose from now that $\tau^n \xrightarrow{BC} \tau$ instead of $\tau^{\varphi(n)} \xrightarrow{BC} \tau$.
We also denote by "continuity points" of a process the points where the process is continuous in probability, ie $t$ such that $\mathbb{P}[\Delta X_t \neq 0] = 0$.

- As $\mathcal{F} \subset \mathcal{G}$, for every $t$, $\forall Z \in L^1(\mathcal{F}_T \times \mathcal{B})$, $\mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t \times \mathcal{B}]$ is $\mathcal{G}_t \times \mathcal{B}$-measurable.
- Let us show $\forall t \in [0,T], \forall Z \in L^1(\mathcal{F}_T \times \mathcal{B}), \forall C \in \mathcal{G}_t \times \mathcal{B}$,

$$\mathbb{E}_{\mathbb{P}\otimes\mu}[\mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t \times \mathcal{B}]1_C] = \mathbb{E}_{\mathbb{P}\otimes\mu}[Z1_C].$$

Let us fix $t \in [0,T]$ and $\varepsilon > 0$.
Let us take $Z \in L^1(\mathcal{F}_T \times \mathcal{B})$. By definition of $\mathcal{G}_t \times \mathcal{B}$, it suffices to prove that for every $A \in \mathcal{F}_t$, for every $s \leqslant t$ and for every $B \in \mathcal{B}$,

$$\begin{aligned} &\int\int_{\Omega\times[0,1]} Z(\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d\mathbb{P}(\omega)dv \\ &= \int\int_{\Omega\times[0,1]} \mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t\times\mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d\mathbb{P}(\omega)dv. \end{aligned} \tag{22}$$

We first prove that (22) holds for $Z = 1_{A_1\times A_2}$, $A_1 \in \mathcal{F}_T$, $A_2 \in \mathcal{B}$.
We can find $l \in \mathbb{N}$, some continuity points of $X$ $s_1 < \ldots < s_l$ and a continuous bounded function $f$ such that

$$\mathbb{E}_{\mathbb{P}}[|1_{A_1} - f(X_{s_1},\ldots,X_{s_l})|] \leqslant \varepsilon. \tag{23}$$

Then

$$\int\int |1_{A_1\times A_2}(\omega,v) - f(X_{s_1}(\omega),\ldots,X_{s_l}(\omega))1_{A_2}(v)|d\mathbb{P}(\omega)dv \leqslant \varepsilon.$$

Let us fix $A \in \mathcal{F}_t$. We can find $k \in \mathbb{N}$, $t_1 < \ldots < t_k \leqslant t$ where $t_i$ are continuity points of $X$ and $H : \mathbb{R}^k \to \mathbb{R}$ bounded continuous such that

$$\mathbb{E}_{\mathbb{P}}[|1_A - H(X_{t_1},\ldots,X_{t_k})|] \leqslant \varepsilon. \tag{24}$$

Let $u \geqslant t$ be a continuity point of $\mathbb{E}[f(X_{s_1},\ldots,X_{s_l})|\mathcal{F}_\cdot]$ and of $\tau^*$.
Fix $s \leqslant t$. We can find $G$ bounded continuous such that

$$\mathbb{E}_{\mathbb{P}\otimes\mu}[|1_{\{\tau^*\leqslant s\}} - G(\tau^*\wedge u)|] \leqslant \varepsilon. \tag{25}$$

$B \in \mathcal{B}$ and the set of continuous functions is dense into $L^1(\mu)$, so there exists $g : \mathbb{R} \to \mathbb{R}$ bounded continuous such that

$$\int |1_B(v) - g(v)|dv \leqslant \varepsilon. \tag{26}$$

We are going to show that

$$\begin{aligned} &\int\int \mathbb{E}_{\mathbb{P}\otimes\mu}[f(X_{s_1},\ldots,X_{s_l})1_{A_2}|\mathcal{F}_u\otimes\mathcal{B}](\omega,v)H(X_{t_1}(\omega),\ldots,X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v) \\ = &\int\int f(X_{s_1}(\omega),\ldots,X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega),\ldots,X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v). \end{aligned}$$

$X^n \xrightarrow{\mathbb{P}} X$, $s_i$ are continuity points of $X$ and $f$ is a bounded continuous function, then

$$f(X^n_{s_1}, \dots, X^n_{s_l}) \xrightarrow{L^1} f(X_{s_1}, \dots, X_{s_l}). \tag{27}$$

Moreover, $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ so using (Coquet, Mémin and Słomiński, 2001, Remark 2),

$$\mathbb{E}_{\mathbb{P}}[f(X^n_{s_1}, \dots, X^n_{s_l})|\mathcal{F}^n] \xrightarrow{\mathbb{P}} \mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}].$$

Since $u$ is a continuity point of $\mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_.]$, we have

$$\mathbb{E}_{\mathbb{P}}[f(X^n_{s_1}, \dots, X^n_{s_l})|\mathcal{F}^n_u] \xrightarrow{\mathbb{P}} \mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u].$$

Since $f$ is bounded, convergence is in $L^1$ :

$$\mathbb{E}_{\mathbb{P}}[f(X^n_{s_1}, \dots, X^n_{s_l})|\mathcal{F}^n_u] \xrightarrow{L^1} \mathbb{E}_{\mathbb{P}}[f(X_{s_1}, \dots, X_{s_l})|\mathcal{F}_u]. \tag{28}$$

Let us consider the maps $\tilde{H}$, $\tilde{G}$ and $\tilde{g}$ from $\mathbb{R}^{k+l+2}$ to $\mathbb{R}$ defined as follows :

$$\begin{aligned} \tilde{H}(x_1, \dots, x_l, y_1, \dots, y_k, z, v) &= H(y_1, \dots, y_k), \\ \tilde{G}(x_1, \dots, x_l, y_1, \dots, y_k, z, v) &= G(z), \\ \tilde{g}(x_1, \dots, x_l, y_1, \dots, y_k, z, v) &= g(v). \end{aligned}$$

Fix $\varepsilon' > 0$.
The set of continuous maps is dense into $L^1(\mathbb{P}_{(X_{s_1},\dots,X_{s_l})} \otimes \mu)$, hence we can find $h : \mathbb{R}^{l+1} \to \mathbb{R}$ such that

$$\int\int |h(x_1, \dots, x_l, v) - f(x_1, \dots, x_l)1_{A_2}(v)| d(\mathbb{P}_{(X_{s_1},\dots,X_{s_l})} \otimes \mu)(\omega, v) \leqslant \varepsilon'. \tag{29}$$

$X^n \xrightarrow{\mathbb{P}} X$, $s_i$ are continuity points of $X$ and $h$ is a bounded continuous function, so

$$\int\int |h(X^n_{s_1}(\omega), \dots, X^n_{s_l}(\omega), v) - h(X_{s_1}(\omega), \dots, X_{s_l}(\omega), v)| d(\mathbb{P} \otimes \mu)(\omega, v) \xrightarrow[n\to+\infty]{} 0. \tag{30}$$

Then we consider :

$$\tilde{h}(x_1, \dots, x_l, y_1, \dots, y_k, z, v) = h(x_1, \dots, x_l, v).$$

$\tilde{h}\tilde{H}\tilde{G}\tilde{g}$ is continuous as product of continuous maps.
Moreover, $(X^n, \tau^{*,n}) \xrightarrow{\mathcal{L}} (X, \tau^*)$ and $u$ is a continuity point of $\tau^*$, so that $(X^n, \tau^{*,n} \wedge u) \xrightarrow{\mathcal{L}} (X, \tau^* \wedge u)$.
Let $U : \Omega \times [0,1] \to [0,1]$ be the random variable such that $\forall\omega, \forall v, U(\omega, v) = v$. As in the proof of Proposition 21, we have :

$$(X^n, \tau^{*,n} \wedge u, U) \xrightarrow{\mathcal{L}} (X, \tau^* \wedge u, U).$$

As $s_1, \dots, s_l, t_1, \dots, t_k$ are continuity points of $X$, we have

$$(X^n_{s_1}, \dots, X^n_{s_l}, X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n} \wedge u, U) \xrightarrow{\mathcal{L}} (X_{s_1}, \dots, X_{s_l}, X_{t_1}, \dots, X_{t_k}, \tau^* \wedge u, U).$$

Hence,

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}\otimes\mu}[(\tilde{h}\tilde{H}\tilde{G}\tilde{g})(X^n_{s_1}, \dots, X^n_{s_l}, X^n_{t_1}, \dots, X^n_{t_k}, \tau^{*,n} \wedge u, U)] \\ &\xrightarrow[n\to+\infty]{} \quad \mathbb{E}_{\mathbb{P}\otimes\mu}[(\tilde{h}\tilde{H}\tilde{G}\tilde{g})(X_{s_1}, \dots, X_{s_l}, X_{t_1}, \dots, X_{t_k}, \tau^* \wedge u, U)]. \end{aligned} \tag{31}$$

By definition of functions $\tilde{h}$, $\tilde{H}$, $\tilde{G}$ and $\tilde{g}$, we have :

$$\begin{aligned} &\int\int h(X^n_{s_1}(\omega), \dots, X^n_{s_l}(\omega), v) H(X^n_{t_1}(\omega), \dots, X^n_{t_k}(\omega)) \\ &\qquad G(\tau^{*,n}(\omega, v) \wedge u) g(v) d(\mathbb{P} \otimes \mu)(\omega, v) \\ \xrightarrow[n\to+\infty]{} &\int\int h(X_{s_1}(\omega), \dots, X_{s_l}(\omega), v) H(X_{t_1}(\omega), \dots, X_{t_k}(\omega)) \\ &\qquad G(\tau^*(\omega, v) \wedge u) g(v) d(\mathbb{P} \otimes \mu)(\omega, v). \end{aligned} \tag{32}$$

Then using triangular inequalities and inequations (27), (30), (32) and (29), we get

$$\begin{aligned}
&\limsup_n \Big| \int\int f(X^n_{s_1}(\omega),\dots,X^n_{s_l}(\omega))1_{A_2}(v)H(X^n_{t_1}(\omega),\dots,X^n_{t_k}(\omega)) \\
&\qquad G(\tau^{*,n}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v) \\
&\quad -\int\int f(X_{s_1}(\omega),\dots,X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega)) \\
&\qquad G(\tau^{*}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v)\Big| \\
\leqslant\ & 2\|H\|_\infty\|G\|_\infty\|g\|_\infty\varepsilon'.
\end{aligned}$$

This is true for every $\varepsilon'>0$, so :

$$\begin{aligned}
&\int\int f(X^n_{s_1}(\omega),\dots,X^n_{s_l}(\omega))1_{A_2}(v)H(X^n_{t_1}(\omega),\dots,X^n_{t_k}(\omega)) \\
&\qquad G(\tau^{*,n}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v) \qquad (33)\\
\xrightarrow[n\to+\infty]{}\ &\int\int f(X_{s_1}(\omega),\dots,X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega)) \\
&\qquad G(\tau^{*}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v).
\end{aligned}$$

On the other hand, $\mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_u\times\mathcal{B}]=\mathbb{E}[f(X_{s_1},\dots,X_{s_l})|\mathcal{F}_u]1_{A_2}$. $\mathbb{E}[f(X_{s_1},\dots,X_{s_l})|\mathcal{F}_u]$ is $\mathcal{F}_u$-measurable.
Let us fix $\varepsilon'>0$. We can find $j\in\mathbb{N}$ and $v_1<\dots<v_j\leqslant u$ some continuity points of $X$ and $F:\mathbb{R}^j\to\mathbb{R}$ bounded continuous such that :

$$\mathbb{E}_{\mathbb{P}}[|\mathbb{E}[f(X_{s_1},\dots,X_{s_l})|\mathcal{F}_u]-F(X_{v_1},\dots,X_{v_j})|]\leqslant\varepsilon'. \qquad (34)$$

$X^n\xrightarrow{\mathbb{P}}X$, $F$ is bounded continuous and $v_i$ are continuity points of $X$ then

$$F(X^n_{v_1},\dots,X^n_{v_j})\xrightarrow{L^1}F(X_{v_1},\dots,X_{v_j}). \qquad (35)$$

As previously, we have :

$$\begin{aligned}
&\int\int F(X^n_{v_1}(\omega),\dots,X^n_{v_j}(\omega))1_{A_2}(v)H(X^n_{t_1}(\omega),\dots,X^n_{t_k}(\omega)) \\
&\qquad G(\tau^{*,n}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v) \\
\xrightarrow[n\to+\infty]{}\ &\int\int F(X_{v_1}(\omega),\dots,X_{v_j}(\omega))1_{A_2}(v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega)) \\
&\qquad G(\tau^{*}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v). \qquad (36)
\end{aligned}$$

Then using triangular inequalities and inequations (27), (28), (34), (35) and (36), we have :

$$\begin{aligned}
&\limsup_n \Big| \int\int \mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}|\mathcal{F}^n_u\times\mathcal{B}](\omega,v)H(X^n_{t_1}(\omega),\dots,X^n_{t_k}(\omega)) \\
&\qquad G(\tau^{*,n}\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v) \\
&\quad -\int\int \mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_u\times\mathcal{B}](\omega,v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega)) \\
&\qquad G(\tau^{*}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v)\Big| \\
\leqslant\ & 2\|H\|_\infty\|G\|_\infty\|g\|_\infty\varepsilon'.
\end{aligned}$$

This is true for every $\varepsilon'>0$, so :

$$\begin{aligned}
&\int\int \mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}|\mathcal{F}^n_u\times\mathcal{B}](\omega,v)H(X^n_{t_1}(\omega),\dots,X^n_{t_k}(\omega)) \\
&\qquad G(\tau^{*,n}\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v) \\
\xrightarrow[n\to+\infty]{}\ &\int\int \mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_u\times\mathcal{B}](\omega,v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega)) \\
&\qquad G(\tau^{*}(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v). \qquad (37)
\end{aligned}$$

But, $H(X^n_{t_1},\dots,X^n_{t_k})$ is $\mathcal{F}^n_u\times\mathcal{B}$-measurable and $G(\tau^n\wedge u)$ and $g(U)$ are also $\mathcal{F}^n_u\times\mathcal{B}$-measurable, by continuity of $G$ and $g$. Then,

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}|\mathcal{F}^n_u\times\mathcal{B}]H(X^n_{t_1},\dots,X^n_{t_k})G(\tau^n\wedge u)g(U)]\\
&\quad=\ \mathbb{E}[\mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}H(X^n_{t_1},\dots,X^n_{t_k})G(\tau^n\wedge u)g(U)|\mathcal{F}^n_u\times\mathcal{B}]]\\
&\quad=\ \mathbb{E}[f(X^n_{s_1},\dots,X^n_{s_l})1_{A_2}H(X^n_{t_1},\dots,X^n_{t_k})G(\tau^n\wedge u)g(U)]
\end{aligned}$$

Using unicity of the limit and convergences (32) and (37), we obtain :

$$\begin{aligned}
&\int\int\mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_u\times\mathcal{B}](\omega,v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega))\\
&\qquad G(\tau^*(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v)\\
=\ &\int\int f(X_{s_1}(\omega),\dots,X_{s_l}(\omega))1_{A_2}(v)H(X_{t_1}(\omega),\dots,X_{t_k}(\omega))\\
&\qquad G(\tau^*(\omega,v)\wedge u)g(v)d(\mathbb{P}\otimes\mu)(\omega,v).
\end{aligned}\tag{38}$$

Then,

$$\begin{aligned}
&\Big|\int\int\mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_u\times\mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\\
&\quad-\int\int f(X_{s_1}(\omega),\dots,X_{s_l}(\omega))1_{A_2}(v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\Big|\\
&\leqslant 2\|f\|_\infty(1+\|H\|_\infty+\|G\|_\infty)\varepsilon\ \text{ using (24), (25), (26) and (38).}
\end{aligned}$$

Let $u$ tend to $t$ by upper values. $\mathbb{E}[f(X_{s_1},\dots,X_{s_l})|\mathcal{F}_.]$ is a càdlàg process, so we have :

$$\begin{aligned}
&\Big|\int\int\mathbb{E}[f(X_{s_1},\dots,X_{s_l})1_{A_2}|\mathcal{F}_t\times\mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\\
&\quad-\int\int f(X_{s_1}(\omega),\dots,X_{s_l}(\omega))1_{A_2}(v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\Big|\\
&\qquad\leqslant 2\|f\|_\infty(1+\|H\|_\infty+\|G\|_\infty)\varepsilon.
\end{aligned}\tag{39}$$

Then,

$$\begin{aligned}
&\Big|\int\int\mathbb{E}[Z|\mathcal{F}_t\times\mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}d(\mathbb{P}\otimes\mu)(\omega,v)\\
&\quad-\int\int Z(\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\Big|\\
\leqslant\ &2\|f\|_\infty(1+\|H\|_\infty+\|G\|_\infty)\varepsilon+2\varepsilon\ \text{ using (23) and (39).}
\end{aligned}$$

This is true for every $\varepsilon>0$, so we have the equality (22) :

$$\begin{aligned}
&\int\int\mathbb{E}[Z|\mathcal{F}_t\times\mathcal{B}](\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v)\\
&\quad=\ \int\int Z(\omega,v)1_A(\omega)1_{\{\tau^*(\omega,v)\leqslant s\}}1_B(v)d(\mathbb{P}\otimes\mu)(\omega,v),
\end{aligned}$$

for every $t\in[0,T]$, for every $Z=1_{A_1\times A_2}$, $A_1\in\mathcal{F}_T$, $A_2\in\mathcal{B}$, for every $A\in\mathcal{F}_t$, for every $s\leqslant t$, for every $B\in\mathcal{B}$.

If $Z=1_E$ with $E\in\mathcal{F}_T\times\mathcal{B}$, (22) holds using the preceding results and an argument of monotone class.

Then, if $Z$ is a function of the form $\sum a_i1_{A_i}$ with $a_i\in\mathbb{R}$ and $A_i\in\mathcal{F}_T\times\mathcal{B}$, (22) holds by linearity.

If $Z$ is $\mathcal{F}_T\times\mathcal{B}$-measurable, we use density in $L^1$ norm of the functions of the form $\sum a_i1_{A_i}$ to obtain (22).

Hence, for every $t \in [0,T]$, for every $Z \in L^1(\mathcal{F}_T \times \mathcal{B})$, for every $C \in \mathcal{G}_t \times \mathcal{B}$ (by definition of $\mathcal{G}_t \times \mathcal{B}$),

$$\mathbb{E}_{\mathbb{P}\otimes\mu}[\mathbb{E}_{\mathbb{P}\otimes\mu}[Z|\mathcal{F}_t \times \mathcal{B}]1_C] = \mathbb{E}_{\mathbb{P}\otimes\mu}[Z1_C].$$

The assumption of Lemma 25 if filled, so

$$\mathbb{E}[\gamma(\tau, Y_\tau)] \leqslant \tilde{\Gamma}(L) = \Gamma^*(L).$$

Using inequality (20), we finally have

$$\limsup \Gamma_n(L) \leqslant \Gamma^*(L).$$

But using Lemma 18, $\Gamma^*(L) = \Gamma(L)$. Theorem 24 is proved. □

To sum up, under the hypothesis of Theorem 2, we have proved the inequality $\Gamma(L) \leqslant \liminf \Gamma_n(L)$ in Theorem 4 and Remark 12. Then, we have shown that $\Gamma(L) \geqslant \liminf \Gamma_n(L)$ when we have the inclusion of filtrations $\mathcal{F}^n \subset \mathcal{F}$ in Theorem 22 and when we have the convergence of filtrations $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ in Theorem 24. Finally, Theorem 2 is proved.

# 7 Applications

## 7.1 Application to discretizations

Let us apply what we have proved in the case of discretizations.

**Proposition 26** *Let us consider a càdlàg process $X$ such that $\mathbb{P}[\Delta X_t \neq 0] = 0$ for every $t$. Let $(\pi^n = \{t_1^n, \dots t_{k^n}^n\})_n$ be an increasing sequence of subdivisions of $[0,T]$ with mesh going to 0 ($|\pi^n| \xrightarrow[n\to+\infty]{} 0$). We define the sequence of discretized processes $(X^n)_n$ by $\forall n$, $\forall t$, $X_t^n = \sum_{i=1}^{k^n-1} X_{t_i^n} 1_{t_i^n \leqslant t < t_{i+1}^n}$.*
*Then $\Gamma_n(L) \xrightarrow[n\to+\infty]{} \Gamma(L)$.*

Proof
Let us consider $\mathcal{F}^X$ the natural filtration for $X$, $\mathcal{F}$ the right continuous associated filtration and $(\mathcal{F}^n)_n$ the natural filtrations for the $(X^n)_n$.
- $X^n \xrightarrow[n\to+\infty]{} X$ *a.s.* then in probability.
- $\forall n, \mathcal{F}^n \subset \mathcal{F}^X \subset \mathcal{F}$ by definition of $X^n$.
- $\mathbb{P}[\Delta X_L \neq 0] = 0$ by hypothesis.
- Using Proposition 15, Aldous' criterion is filled.
Then using Theorem 2, $\Gamma_n(L) \xrightarrow[n\to+\infty]{} \Gamma(L)$. □

## 7.2 Application to financial models

We are going to apply the previous results to financial models. For a study about those models, see for example the book (Lamberton and Lapeyre, 1997).

We wish to find the price of an American call option at the best time of exercise for the buyer. We denote by $T$ the maturity date of this call option. The market is composed of an asset with risk of price $S_t$ at time $t$ and an asset without risk of price $S_t^0$ at time $t$. We assume that $S_t$ follows the stochastic differential equation $dS_t = S_t(\mu dt + \sigma dB_t)$ where $\mu$ and $\sigma$ are positive reals and $(B_t)$ is a standart brownian motion. We also assume that $S_t^0$ is solution of the ordinary differential equation $dS_t^0 = rS_t^0 dt$ where $r > 0$.

We define the actualized price of the asset with risk by $\tilde{S}_t = e^{-rt}S_t$. Then, we have $d\tilde{S}_t = \tilde{S}_t(\lambda dt + \sigma dB_t)$ where $\lambda = \mu - r$. The solution of this equation is well known :

$\tilde{S}_t = \tilde{S}_0 exp(\lambda t - \sigma^2 t/2 + \sigma B_t)$.

The natural filtration for $\tilde{S}$ is the brownian filtration, denoted by $\mathcal{F}$. At the optimal exercice date, the price of the option is given by the following value in optimal stopping of horizon $T$ for $\tilde{S}$ :

$$\Gamma^{\tilde{S}}(T) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[\tilde{S}_\tau],$$

where $\mathcal{T}$ is the set of $\mathcal{F}$ stopping times bounded by $T$.

It is usual to approximate the model of Black and Scholes by a sequence of models of Cox-Ross-Rubinstein.

On an adapted space, we consider a sequence $(X_i)$ of independent Bernoulli variables such that $\forall i, \mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$. For every $n \in \mathbb{N}^*$, we consider $B^n_{kT/n} = \sqrt{T/n}\sum_{i=1}^k X_i, k = 0, \ldots, n$. We assume that the actualized prices $\tilde{S}^n_{kT/n}$ of the asset with risk at time $kT/n$ are given by the linear equation $\Delta\tilde{S}^n_{(k+1)T/n} = \tilde{S}^n_{kT/n}(\lambda_n T/n + \sigma_n \Delta B^n_{(k+1)T/n})$ where $\Delta\tilde{S}^n_{(k+1)T/n} = \tilde{S}^n_{(k+1)T/n} - \tilde{S}^n_{kT/n}$ and $\Delta B^n_{(k+1)T/n} = B^n_{(k+1)T/n} - B^n_{kT/n}$.

We extend processes $B^n$ and $\tilde{S}^n$ to $[0,T]$ by the following : $B^n_t = B^n_{kT/n}$ if $kT/n \leqslant t < (k+1)T/n$ and $\tilde{S}^n_t = \tilde{S}^n_{kT/n}$ if $kT/n \leqslant t < (k+1)T/n$.

The natural filtration for $\tilde{S}^n$ is $\mathcal{F}^n$ such that $\mathcal{F}^n_t = \sigma(B^n_{kT/n}, kT/n \leqslant t)$, for every $t$. At the optimal exercise date, the value of the option is given by the following reduite of horizon $T$ associated to $\tilde{S}^n$ :

$$\Gamma^{\tilde{S}^n}(T) = \sup_{\tau \in \mathcal{T}^n} \mathbb{E}[\tilde{S}^n_\tau],$$

where $\mathcal{T}^n$ is the set of $\mathcal{F}^n$ stopping times bounded by $T$.

We assume that $\lambda_n \xrightarrow[n\to+\infty]{} \lambda$ and $\sigma_n \xrightarrow[n\to+\infty]{} \sigma$.

Using Donsker's Theorem, we have :

$$(B^n, \tilde{S}^n) \xrightarrow{\mathcal{L}} (B, \tilde{S}).$$

According to the Skorokhod representation Theorem, we can find processes $(X, Y)$ and $((X^n, Y^n))_n$ such that $\forall n$, $(X^n, Y^n) \sim (B^n, \tilde{S}^n)$, $(X, Y) \sim (B, \tilde{S})$ and $(X^n, Y^n) \xrightarrow{a.s.} (X, Y)$.

But, $\tilde{S}^n$ is a continuous function of $B^n$ and $(X^n, Y^n) \sim (B^n, \tilde{S}^n)$ so $Y^n$ is a continuous function of $X^n$. Hence, $Y^n$ and $X^n$ have the same natural filtration $\mathcal{F}^{X^n} = \mathcal{F}^{Y^n}$. Similarly, $X$ and $Y$ have the same natural filtration $\mathcal{F}^X = \mathcal{F}^Y$.

Moreover, $B$ is a process with independent increments, so also is $X$. Then, using (Coquet, Mémin and Słomiński, 2001, Theorem 2), as $X^n \xrightarrow{\mathbb{P}} X$, we have the corresponding convergence of filtrations : $\mathcal{F}^{X^n} \xrightarrow{w} \mathcal{F}^X$. Hence, $\mathcal{F}^{Y^n} \xrightarrow{w} \mathcal{F}^Y$.

$Y$ and $\tilde{S}$ have the same law so $Y$ is quasi-left continuous. $Y^n \xrightarrow{\mathbb{P}} Y$, $\mathcal{F}^{Y^n} \xrightarrow{w} \mathcal{F}^Y$ and $Y$ is quasi-left continuous, so using Theorem 2, we have

$$\Gamma^{Y^n}(T) \xrightarrow[n\to+\infty]{} \Gamma^Y(T)$$

where $\Gamma^{Y^n}(T) = \sup_{\tau \in \mathcal{T}^{Y^n}} \mathbb{E}[Y^n_\tau]$ with $\mathcal{T}^{Y^n}$ the set of $\mathcal{F}^{Y^n}$ stopping times bounded by $T$ and $\Gamma^Y(T) = \sup_{\tau \in \mathcal{T}^Y} \mathbb{E}[Y_\tau]$ with $\mathcal{T}^Y$ the set of $\mathcal{F}^Y$ stopping times bounded by $T$.

But according to Remark 1, the value in optimal stopping only depends on the law of the process. Here, $Y$ and $\tilde{S}$ have the same law so $\Gamma^{Y}(T)=\Gamma^{\tilde{S}}(T)$ and $Y^n$ and $\tilde{S}^n$ have the same law so $\Gamma^{Y^n}(T)=\Gamma^{\tilde{S}^n}(T)$. Then, the sequence of values in optimal stopping associated to the models of Cox-Ross-Rubinstein converges to the value in optimal stopping of the model of Black and Scholes :

$$\Gamma^{\tilde{S}^n}(T) \xrightarrow[n\to+\infty]{} \Gamma^{\tilde{S}}(T).$$

## References


Aldous, D., 1978. Stopping times and tightness. Ann. Proba., 6(2):335–340.

Aldous, D., 1989. Stopping times and tightness. II. Ann. Proba., 17(2):586–595.

Aldous, D., 1981. Weak convergence of stochastic processes for processes viewed in the Strasbourg manner. Preprint, Statis. Laboratory Univ. Cambridge.

Baxter, J.R., Chacon, R.V., 1977. Compactness of stopping times. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 40 (3):169–181.

Billingsley, P., 1999. Convergence of Probability Measures, Second Edition. Wiley and Sons, New York.

Brémaud, P., Yor, M., 1978. Changes of filtrations and of probability measures. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 45:269–295.

Coquet, F., Mémin, J., Słomiński, L., 2001. On weak convergence of filtrations. Séminaire de probabilités XXXV, Lectures Notes in Mathematics, Springer Verlag, Berlin Heidelberg New York, 1755:306–328.

Hoover, D.N. , 1991. Convergence in distribution and Skorokhod convergence for the general theory of processes. Probab. Theory Related Fields, 89(3):239–259.

Jacod, J., Mémin, J., 1981. Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité. Séminaire de Probabilités, XV, Lectures Notes in Mathematics, Springer, Berlin, 850:529–546.

Jacod, J., Shiryaev, A.N., 2002. Limit Theorems for Stochastic Processes, Second Edition. Springer Verlag, Berlin Heidelberg New York.

Lamberton, D., Lapeyre, B., 1997. Introduction au calcul stochastique appliqué à la finance, Seconde Edition. Ellipses Edition Marketing, Paris.

Lamberton, D., Pagès, G., 1990. Sur l'approximation des réduites. Ann. Inst. Henri Poincaré, 26(2):331–355.

Meyer, P.A., 1978. Convergence faible et compacité des temps d'arrêt d'après Baxter et Chacon. Séminaire de Probabilités, XII, Lectures Notes in Mathematics, Springer, Berlin, 649:411–423.

Renyi, A., 1963. On stable sequences of events. Sankya, Ser A, 25:293–302.